Generalized Phi Number System and its Applications for Image Decomposition and Enhancement

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ABSTRACT

Technologies and applications of the field-programmable gate array (FPGAs) and digital signal processing (DSP) require both new customizable number systems and new data formats. This paper introduces a new class of parameterized number systems, namely the generalized Phi number system (GPNS). By selecting appropriate parameters, the new system derives the traditional Phi number system, binary number system, beta encoder, and other commonly used number systems. GPNS also creates new opportunities for developing customized number systems, multimedia security systems, and image decomposition and enhancement systems. A new image enhancement algorithm is also developed by integrating the GPNS-based bit-plane decomposition with Parameterized Logarithmic Image Processing (PLIP) models. Simulation results are given to demonstrate the GPNS's performance.

Keywords: Number system, Rational number system, Phi number system, Parameterized Logarithmic Image Processing, Bit-plane decomposition, Image enhancement

1. INTRODUCTION

The development of computer technologies is how to utilize the hardware and software to perform computation. Digital computer arithmetic is based on data or numbers represented as a string of zeroes and ones. The hardware of digital computers can execute the simple and primitive formats of Boolean operations. The size of the digits is called radix *r*:

$$X = a_k r^k + a_{k-1} r^{(k-1)} + ... + a_1 r + a_0 \tag{1}$$

The radix *r* could be an integer, positive or a constant. A very useful radix case is r=2, also called the base-2 number system. It represents numbers by a combination of two numerals: $a_k \in \{0,1\}$, zero (0) and one (1). All modern computers are based on the binary system. The computer architecture is from embedded processors to supercomputers. The binary system is also used for parallel programming and parallel architectures, computer arithmetic, custom computing, algorithms and programming methodologies as well. Binary arithmetic operations are implemented by Boolean logics which are easily realized with digital electronics. In other words, binary number system was selected for computer systems because of its straightforward and one-to-one mapping in logic circuits.

Due to the long carry/borrow propagation paths extending from the least-significant bit to the most-significant bit position, the binary number systems are limited in the computing speed [1, 2]. This motivates researchers to develop alternative approaches to design high-speed arithmetic units. Using unconventional number representations has brought much attention in recent years. Using non-binary number representations, several schemes have been presented in order to obtain efficient arithmetic operations. Examples include the multiple-valued fixed radix-number residue number [3], Logarithmic Number System (LNS) [4, 5], Residue Number System (RNS) [6, 7], signed-digit [8], and some hybrid number systems [9, 10]. An appropriate number representation is very important for different FPGA applications since it significantly affects the performance and accuracy of the FPGA systems. Recently several encoders have been developed for different number systems such as rational number system [5], beta or Golden ratio encoders [11, 12] which are used on analog to digital (A/D) conversion system [13].

The golden ratio ($\varphi = (1+\sqrt{5})/2$) sometimes refers to base-φ (the Greek letter "phi"). It has been used as the base of a number system called golden mean base, phi number system (PNS), or phinary. This number system was developed by Bergman in 1957. It is a non-integer positional number system in which the base-φ is an irrational number.

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Every non-negative real number can be represented as a base-φ sequence using digits 0 and 1, and avoiding the digit sequence “11” – this called *a standard form* [14-16]. In this number system, every non-negative integer is represented by a finite base-φ sequence of distinct integers $k_1, k_2, ..., k_m$ such that [15, 17-19].

$$X = \varphi^{k_1} + \varphi^{k_2} + ... + \varphi^{k_m} \tag{2}$$

Every non-negative integer has a base-φ representation using two digits 0 and 1. This representation is unique if the following condition is satisfied [14, 15].

1) It contains no consecutive 1s or has a *standard form*.
2) It does not become “01010101…” after some finite number of digits.

Using the arithmetic properties of the base-φ that $\varphi+1 = \varphi^2$, a base-φ numeral containing the digit sequence "11" can always be rewritten as a standard form. For example, $11_\varphi = 100_\varphi$.

Due to the fact of that all integers have a unique representation as a finite expansion of the irrational base-φ, the phi number system is a remarkable number system. Other numbers have unique standard representations in base-φ as well. Rational numbers have recurring expansions. The numbers with a terminating expansion also have a non-terminating representation [20]. Similar cases could be found in base-10 system, for example, 1=0.9999….[21]. This property can be extended into the negative of a base-φ representation by negating each digit, standardizing the result, and then marking it as negative. For example, use a minus sign, or some other significance to denote negative numbers. If the arithmetic is being performed on a computer, an error message may be returned.

Detail discussions of the phi number system can be found in [16, 19, 22]. Ternary τ-representation [23, 24] is another irrational base number system which the base is the square of the “golden ratio”, $\varphi^2 = (3+\sqrt{5})/2$, with the ternary numerals $\{-1, 0, 1\}$. This number system have been used in computers for arithmetical operation control [23, 24]

FPGA technologies make application-specific number representations available with customizable bit-widths. However, they require a new “customized” number system and new data treatment [10, 25]. In this paper, we introduce a new generalized Phi number system (GPNS). By selecting appropriate parameters, The GPNS can be specified to the traditional PNS, the binary number system (base-2), and other integer base number systems. We investigate the applications of the new GPNS in image processing. We introduce a new parameter bit-plane decomposition method using the new GPNS. Integrating this new decomposition method with the chaotic logistic map, a new image encryption algorithm is introduced. Experimental results are given to demonstrate that the GPNS shows excellent performance in image decomposition and encryption.

The rest of the paper is organized as followed. Section 2 introduces the new GPNS and discusses its properties. Section 3 introduces an image bit-plane decomposition method using the GPNS. Section 4 draws a conclusion.

2. GENERALIED PHI NUMBER SYSTEM

In this section, we introduce a new class of parameterized number system, namely the generalized phi number system (GPNS). We show that, by selecting appropriate parameters, the new system can derive other commonly used number systems including the Phi number system, binary number system, and beta encoder. Some GPNS properties are also presented.

By extending the concept of the base-φ representation, we introduce a new number system defined in the definition 2.1.

Definition 2.1: If ψ is a root of $x^2 - bx - c = 0$ (where $b, c > 0$), $\psi = \frac{b + \sqrt{b + 4bc}}{2b}$, each non-negative integer X can be represented by a finite sum,

$$X = a_k \psi^{km} + a_{k-1}\psi^{(k-1)m} + a_{k-2}\psi^{(k-2)m} + ... + a_2\psi^{2m} + a_1\psi^{m} + a_0 + a_{-1}\psi^{-m} + a_{-2}\psi^{-2m} + a_{-3}\psi^{-3m} + ... \tag{3}$$

where a_i, m, k are integers and $m > 0$, $i = 0, 1, 2, 3, ...$. $a_i \in (-\psi, \psi)$. This is called the generalized phi number system (GPNS).

The new GPNS has following properties:

1) The sequence/coefficient a_k can be calculated by using the following iteration

$$
\begin{aligned}
&v_1 = \psi X;\ a_1 = Q(v_1);\\
&i \geq 1:\ v_{i+1} = \psi(v_i - a_i);\\
&\text{for}\quad a_{i+1} = Q(v_{i+1})\\
&\qquad \text{while}\ \ Q(v) = \begin{cases} -1 & v \leq 0 \\ 1 & v > 0 \end{cases}
\end{aligned}
$$

2) The new GPNS is either a rational or irrational number system. The coefficients of $x^2 - bx - c = 0$ determine that the base-ψ is a rational or irrational number.

3) It can be shown that

$$\left| X - \sum_{k=1}^{N} a_k \psi^{km} \right| = \left| \sum_{k=N+1}^{\infty} a_k \psi^{km} \right| \leq \frac{\psi^{-(N+1)m}}{1-\psi^{km}} \quad \text{if} \ \ \psi > 1, \ \text{and} \ m > 0 \tag{4}$$

$$\sum_{k=0}^{\infty} \psi^{km} = \sum_{k=0}^{\infty} \psi^{k2} = \frac{1}{1-\psi^{-2}} \quad \text{if} \ \ \psi > 1, \ \text{and} \ m = 2 \tag{5}$$

$$\sum_{k=0}^{\infty} \psi^{k2} = \frac{1}{1-\psi^{-2}} = \psi \quad \text{if} \ \ \psi = \frac{1+\sqrt{5}}{2} \tag{6}$$

4) By varying its coefficients, b, c, m, a_i ,the GPNS be specified into many well-known number systems, including traditional phi number system, binary numeral system, ternary numeral systems and many other arbitrary base systems.

- *Binary number system*

If $b = 1, c = 2, m = 1,$ and $a_i \in \{0,1\}$, then $x^2 - x - 2 = 0,\ \psi = 2$. The GPNS turns into the traditional binary numeral system defined by,

$$X = a_k 2^k + a_{k-1} 2^{k-1} + a_{k-2} 2^{k-2} + \ldots + a_2 2^2 + a_1 2^1 + a_0 2^0 + a_{-1} 2^{-1} + a_{-2} 2^{-2} + a_{-3} 2^{-3} + \ldots \tag{7}$$

where $a_i \in \{0,1\}$, and $i = 0,1,2,3,\ldots,k$.

The binary numeral system is a base-2 number system. It is widely used in many fields such as digital electronics, digital communications, and all modern computers.

- *Phi number system*

If $b = 1, c = 1, m = 1,$ and $a_i \in \{0,1\}$, then $x^2 - x - 1 = 0,\ \psi = \frac{1+\sqrt{5}}{2}$, the equation (3) changes to

$$X = a_k \psi^k + a_{k-1} \psi^{k-1} + a_{k-2} \psi^{k-2} + \ldots + a_2 \psi^2 + a_1 \psi + a_0 + a_{-1} \psi^{-1} + a_{-2} \psi^{-2} + a_{-3} \psi^{-3} + \ldots \tag{8}$$

The GPNS becomes the traditional phi number system. The equation (8) is another format of the equation (2).

- *Ternary τ-representation*

If $b = 1, c = 1, m = 2,$ and $a_i \in \{-1,0,1\}$, then $x^2 - x - 1 = 0,\ \psi^2 = \frac{3+\sqrt{5}}{2} \approx 2.618$. The GPNS is the ternary τ-representation [23, 24] defined by,

$$X = a_k \psi^{2k} + a_{k-1} \psi^{2(k-1)} + a_{k-2} \psi^{2(k-2)} + \ldots + a_2 \psi^4 + a_1 \psi^2 + a_0 + a_{-1} \psi^{-2} + a_{-2} \psi^{-4} + a_{-3} \psi^{-6} + \ldots \tag{9}$$

where $a_i \in \{-1,0,1\}$, and $i = 0,1,2,3,...,k$.

The ternary τ-representation is the ternary symmetrical number system with the ternary numerals $\{-1,0,1\}$. It has irrational base – the square of the “golden ratio”. It can be used in computers for arithmetical operation control [23, 24].

❖ *Ternary numeral system*

If $b = 1, c = 6, m = 1,$ and $a_i \in \{0,1,2\}$, then $x^2 - x - 6 = 0,\ \psi = 3$. The GPNS is the ternary numeral system – a base-3 number system [26, 27]. It is defined by,

$$X = a_k 3^k + a_{k-1} 3^{k-1} + a_{k-2} 3^{k-2} + ... + a_2 3^2 + a_1 3^1 + a_0 3^0 + a_{-1} 3^{-1} + a_{-2} 3^{-2} + a_{-3} 3^{-3} + ... \quad (10)$$

where $a_i \in \{0,1,2\}$, and $i = 0,1,2,3,...,k$.

❖ *Balanced ternary numeral system*

If $b = 1, c = 6, m = 1,$ and $a_i \in \{-1,0,1\}$, then $x^2 - x - 6 = 0,\ \psi = 3$. The GPNS is the balanced ternary numeral system [28], It is defined by,

$$X = a_k 3^k + a_{k-1} 3^{k-1} + a_{k-2} 3^{k-2} + ... + a_2 3^2 + a_1 3^1 + a_0 3^0 + a_{-1} 3^{-1} + a_{-2} 3^{-2} + a_{-3} 3^{-3} + ... \quad (11)$$

where $a_i \in \{-1,0,1\}$, and $i = 0,1,2,3,...,k$.

The balanced ternary numeral system is also called the ternary symmetrical number system [23, 29]. It is used in comparison logic and ternary computers.

Table 1. GPNS representation of integers.

GPNS	$b=1, c=2, m=1,$ $a_i \in \{0,1\}$	$b=1, c=1, m=1,$ $a_i \in \{0,1\}$	$b=1, c=1, m=2,$ $a_i \in \{-1,0,1\}$	$b=1, c=6, m=1,$ $a_i \in \{0,1,2\}$	$b=1, c=6, m=1,$ $a_i \in \{-1,0,1\}$
	Binary number system	Phi number system	Ternary τ-representation	Ternary numeral system	Balanced ternary numeral system
1	1	1.0	1.0	1	1
2	10	10.01	1 Ī.1	2	1 Ī
3	11	100.01	10.1	10	10
4	100	101.01	11.1	11	11
5	101	1000.1001	1 Ī1. Ī1	12	1 Ī Ī
6	110	1010.0001	10 Ī.01	20	1 Ī0
7	111	10000.0001	100.01	21	1 Ī1
8	1000	10001.0001	101.01	22	10 Ī
9	1001	10010.0101	11 Ī.11	100	100
10	1010	10100.0101	110.11	101	101
11	1011	10101.0101	111.11	102	11 Ī
12	1100	100000.101001	1 Ī01.0 Ī1	110	110
13	1101	100010.001001	1 Ī1 Ī.1 Ī1	111	111
14	1110	100100.001001	1 Ī10.1 Ī1	112	Ī Ī Ī
15	1111	100101.001001	1 Ī11.1 Ī1	120	Ī Ī0
16	10000	101000.100001	10 Ī1. Ī01	121	Ī Ī1
17	10001	101010.000001	100 Ī.001	122	Ī0 Ī
18	10010	1000000.000001	1000.001	200	Ī00
19	10011	1000001.000001	1001.001	201	Ī01
20	10100	1000010.010001	101 Ī.101	202	Ī1 Ī

Note: “Ī” indicates “-1” in the table.

- *Examples of new number systems*

  Case #1: $b = 1, c = \frac{1}{2}, \psi = \frac{1+\sqrt{3}}{2}$

  Case #2: $b = 3, c = 1, \psi = \frac{1+\sqrt{7}}{2}$

  Case #3: $b = 7, c = 1, \psi = \frac{1+\sqrt{11}}{2}$

Table I shows the different representations of integers from 1 to 20.

3. GPNS BIT-PLANE DECOMPOSITION

Benefited from the GPNS properties, we introduce a new GPNS-based image bit-plane decomposition.

The traditional binary bit-plane decomposition [30] intends to decompose image into several binary bit-planes. Each bit-plane contains the corresponding bits of the binary representation of all image pixels. For example, a grayscale image can be decomposed into eight binary bit-planes. The 4th bit-plane consists of the 4th bits of all image pixels. The traditional binary bit-plane decomposition has been used in image processing such as edge detection [31], image coding and compression [32-34], as well as for security applications such as image encryption [35-37], data hiding [38, 39], watermarking [40] and steganography [41, 42].

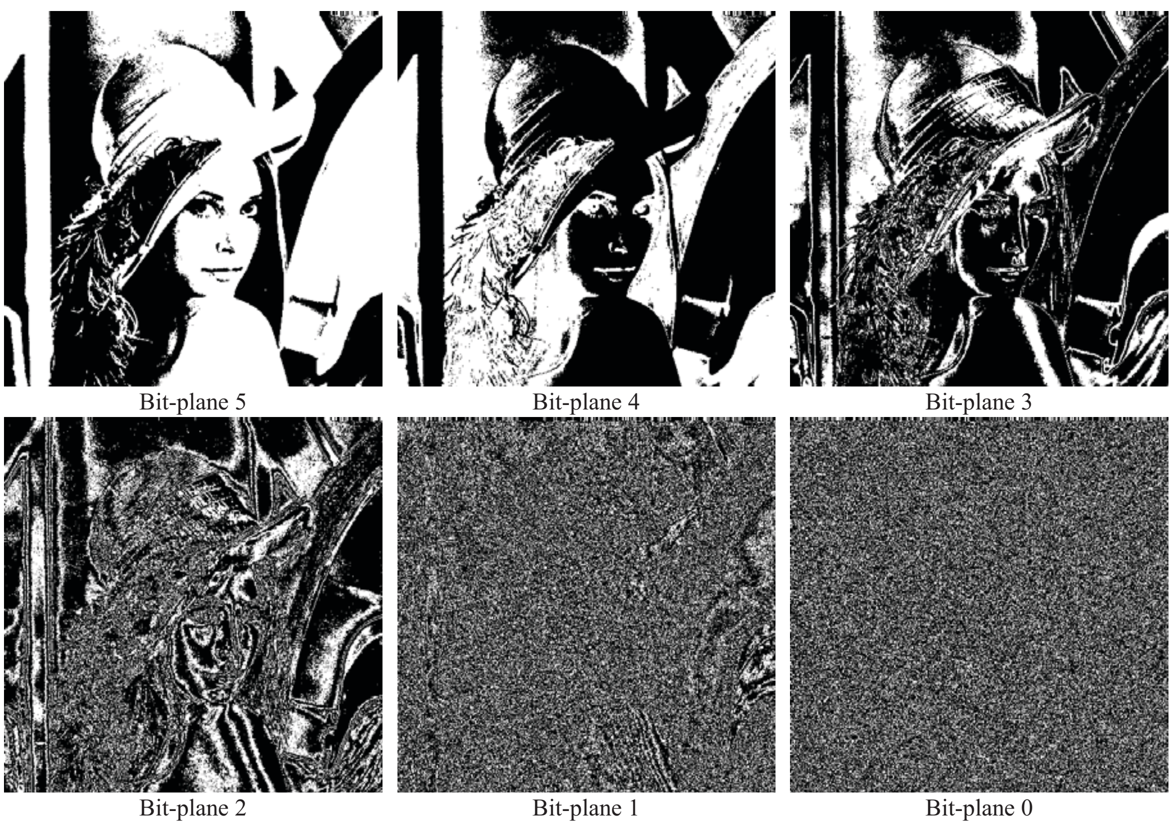


Fig. 1. Image bit-plane decomposition using the balance ternary numeral system (base-3) with numerals $\{-1, 0, 1\}$.

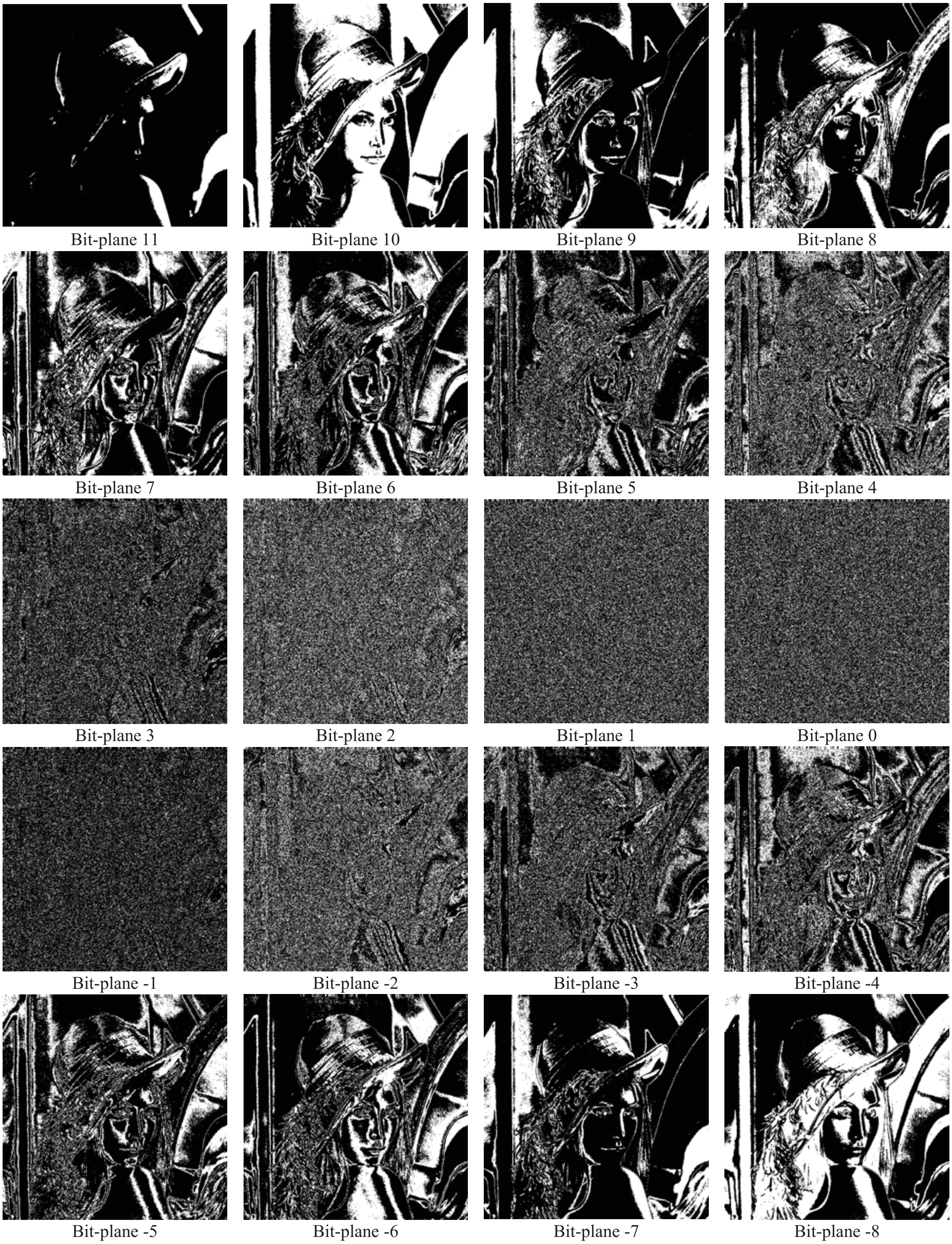

Bit-plane 11
Bit-plane 10
Bit-plane 9
Bit-plane 8
Bit-plane 7
Bit-plane 6
Bit-plane 5
Bit-plane 4
Bit-plane 3
Bit-plane 2
Bit-plane 1
Bit-plane 0
Bit-plane -1
Bit-plane -2
Bit-plane -3
Bit-plane -4
Bit-plane -5
Bit-plane -6
Bit-plane -7
Bit-plane -8

Fig. 2. Image bit-plane decomposition using the Phi number system (base-φ) with numerals $\{0,1\}$.

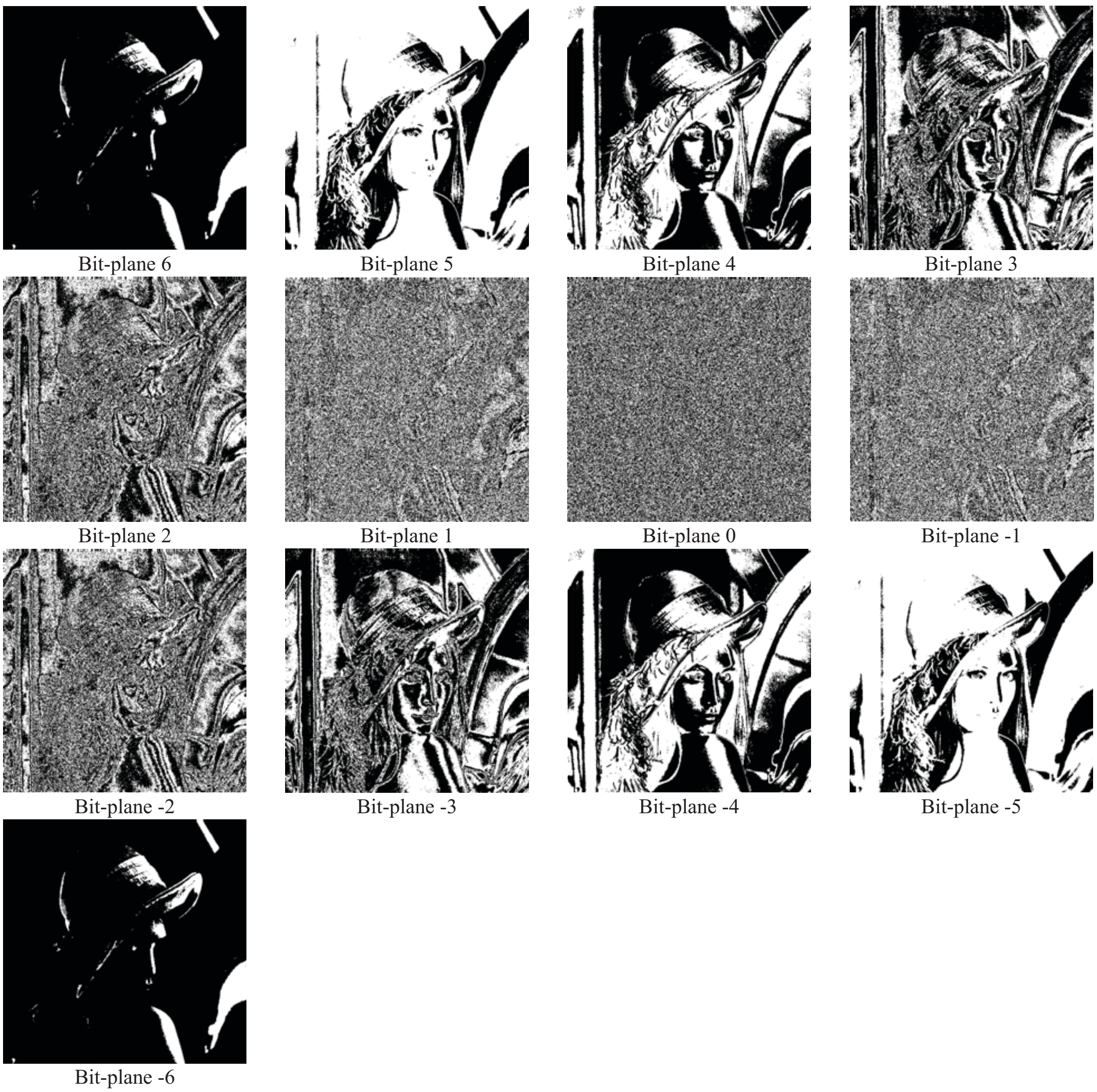


Fig. 3. Image bit-plane decomposition using the ternary τ-representation (base-φ^2) with numerals $\{-1,0,1\}$.

The pixel intensity values of a digital image are non-negative integers. In the same manner of binary bit-plane decomposition, a digital image can also be composed into several GPNS bit-planes. This is called GPNS bit-plane decomposition. Since the base of the GPNS could be an irrational number, a rational number, or an integer. Therefore the GPNS bit-planes may consist of binary bits or arbitrary-base bits. Moreover, the traditional binary bit-plane decomposition is a special case of the GPNS bit-plane decomposition when $b=1, c=2, m=1,$ and $a_i \in \{0,1\}$.

Fig.1-3 provides decomposition results of a grayscale Lena image using the GPNS bit-plane decomposition. Fig. 1 shows image decomposition using the balance ternary numeral system (base-3) with numerals $\{-1,0,1\}$. Using the Phi number system (base-φ) with numerals $\{0,1\}$, the image is decomposed into twenty-five bit-planes shown in Fig. 2. Using the ternary τ-representation (base-φ^2) with numerals $\{-1,0,1\}$, Fig. 3 demonstrates that an image is decomposed into thirteen bit-planes.

4. IMAGE ENHANCEMENT USING GPNS-BASED BIT-PLANE DECOMPOSITION

For a specific base of the GPNS, an image can be decomposed into a certain number of bit-planes with specific numerals. This decomposition result could be used for other applications in image processing. This section investigates the application of the GPNS bit-plane decomposition for image enhancement. A new image enhancement algorithm is introduced. Some results of image enhancement are also presented.

4.1 New Image Enhancement Algorithm

The Parameterized Logarithmic Image Processing (PLIP) model is a mathematical framework [43]. Its operations make use of a parameterized gray-tone function $g(i,j)$ defined by,

$$g(i,j) = \mu(M) - f(i,j) \tag{12}$$

where $f(i,j)$ is the original image intensity.

The multiplication of two gray-tone images, g_1, g_2 in the PLIP model is defined by,

$$g_1 \tilde{*} g_2 = \tilde{\varphi}^{-1}\left(\tilde{\varphi}(g_1)\cdot\tilde{\varphi}(g_2)\right) \tag{13}$$

where $\tilde{\varphi}$ transform and its inverse transform are defined by

$$\tilde{\varphi}(g) = -\lambda(M)\cdot\ln^{\beta}\left(1-\frac{g}{\lambda(M)}\right) \quad \text{and} \quad \tilde{\varphi}^{-1}(g) = \lambda(M)\cdot\left(1-\exp\left(\frac{-g}{\lambda(M)}\right)^{1/\beta}\right)$$

The PLIP multiplication has been demonstrated to be able to yield visually appealing images [43].

Integrating the PLIP multiplication with the GPNS bit-plane decomposition, we introduce a new algorithm for image enhancement, called . The algorithm is shown in Fig. 4.

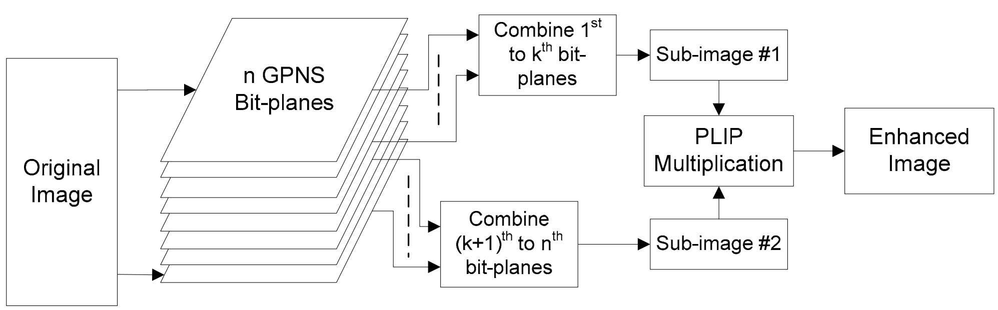


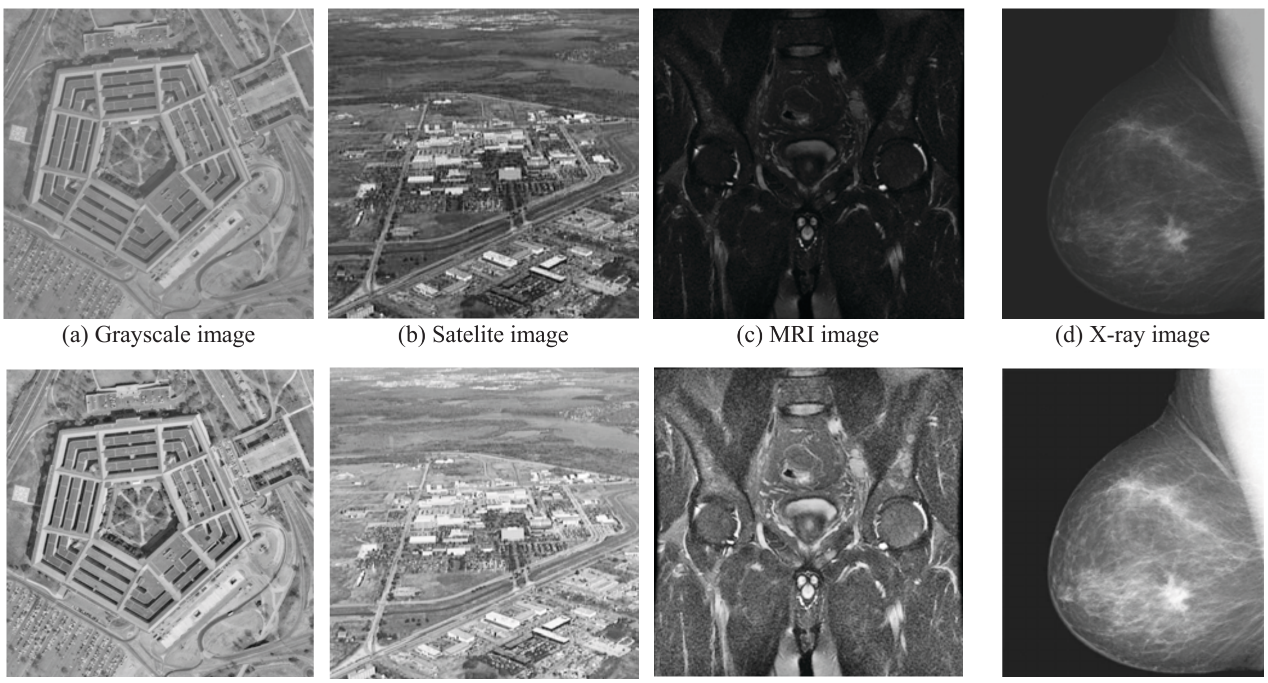
Fig. 4. Image Enhancement Algorithm using the GPNS bit-plane decomposition and PLIP Multiplication

The new enhancement algorithm first decomposes the input image into n GPNS bit-planes with a specific base-ψ. Using the definition 2.1, the algorithm combines the first k GPNS bit-planes to obtain the sub-image #1. The sub-image #2 is generated from the rest GPNS bit-planes by the same manner. The enhanced image is obtained by merging these two sub-images using the PLIP multiplication.

As shown in equations (8) and (9), the variation of parameters, $\mu(M)$, $\lambda(M)$, and β, will change the enhancement results of the presented algorithm by modifying the properties of the PLIP multiplication. When $\lambda(M)$ is negative, the parameter β will play important role for the PLIP multiplication. The new algorithm has potential for enhancing different types of images such as grayscale images, biometrics and medical images.

4.2 Simulation Results

The presented algorithm has been applied to more than 26 images. Fig. 5 shows the enhanced results of four different types of images. The enhanced images are more visually pleasuring and recognizable in details than the original ones. This demonstrates that the new algorithm shows excellent performance for enhancing the contrast and details regions in different types of images.



(a) Grayscale image (b) Satelite image (c) MRI image (d) X-ray image

(e) (f) (g) (h)

Fig. 5. Image enhancement using the presented algorithm. (a)-(d) shows the original images; (e)-(h) shows the corresponding enhanced images. This demonstrates that the presented algorithm can enhance different types of images.

5. CONCLUSION

This paper has introduced a new generalized phi number system, which has shown the ability of driving several existing irrational-, rational- and integer-base number systems. Its applications for image decomposition and enhancement have been investigated. We introduced a new parametric image bit-plane decomposition method using the new GPNS. It can decompose an image into different number of bit-planes according the specification by parameters. Combining this decomposition method with the PLIP multiplication, we have introduced a new image enhancement algorithm. The algorithm has shown excellent performance for improving visual quality of different types of images.

REFERENCES

[1] Johnson, E. L., *Digital design: a pragmatic approach,* PWS Publishing Co., Boston, MA, USA (1987).

[2] Hwang, K., *Computer arithmetic: principles, architecture and design,* John Wiley & Sons, Inc. , New York, NY, USA (1979).

[3] Hurst, S. L., "Multiple-Valued Logic--its Status and its Future," *IEEE Trans. Comput.,* C-33(12), 1160-1179(1984).

[4] Mirsalehi, M. M. and Gaylord, T. K., "Logical minimization of multilevel coded functions," *Appl. Opt.,* 25(18), 3078-3088(1986).

[5] Fu, H., et al., "Optimizing Logarithmic Arithmetic on FPGAs," *Field-Programmable Custom Computing Machines, 2007. FCCM 2007. 15th Annual IEEE Symposium on*, 163-172(2007).

[6] Garner, H. L., "The residue number system," in *Papers presented at the the March 3-5, 1959, western joint computer conference* ACM, San Francisco, California 146-153(1959).

[7] Soderstrand, M., et al., *Residue number system arithmetic: modern applications in digital signal processing,* (1986).

[8] Balakrishnan, W. and Burgess, N., "Very-high-speed VLSI 2s-complement multiplier using signed binary digits," *Computers and Digital Techniques, IEE Proceedings E,* 139(1), 29-34(1992).

[9] Hashemian, R. and Sreedharan, B., "A hybrid number system and its application in FPGA-DSP technology," *Information Technology: Coding and Computing, 2004. Proceedings. ITCC 2004. International Conference on*, 342-346(2004).

[10] Bajard, J.-C., et al., "Modular Number Systems: Beyond the Mersenne Family," in *Selected Areas in Cryptography*, Handschuh and Hasan, Eds. Springer Berlin / Heidelberg, 159-169(2005).

[11] Ward, R., "On Robustness Properties of Beta Encoders and Golden Ratio Encoders," *IEEE Trans. Inf. Theory,* 54(9), 4324-4334(2008).

[12] Daubechies, I., et al., "The Golden Ratio Encoder," *IEEE Trans. Inf. Theory,* 56(10), 5097-5110(2010).

[13] Daubechies, I. and Yilmaz, O., "Robust and Practical Analog-to-Digital Conversion With Exponential Precision," *IEEE Trans. Inf. Theory,* 52(8), 3533-3545(2006).

[14] Bergman, G., "A Number System with an Irrational Base," *Mathematics Magazine,* 31(2), 98-110(1957).

[15] Rousseau, C., "The Phi Number System Revisited," *Mathematics Magazine,* 68(4), 283-284(1995).

[16] Wikipedia. *Golden ratio base*. Available from: http://en.wikipedia.org/wiki/Golden_ratio_base.2010.

[17] Knuth, D. E., *Art of Computer Programming, Volume I: Fundamental Algorithms*, 3 ed. Addison-Wesley Professional (1997).

[18] Levasseur, K. *The Phi Number System*. Available from: http://www.hostsrv.com/webmaa/app1/MSP/webm1010/PhiNumberSystem/PhiNumberSystem.msp.1995.

[19] Weisstein, E. W. *Phi Number System*. Available from: http://mathworld.wolfram.com/PhiNumberSystem.html.

[20] wordiQ. *Golden mean base - Definition* Available from: http://www.wordiq.com/definition/Golden_mean_base.2010.

[21] Wikipedia. *0.999...* Available from: http://en.wikipedia.org/wiki/Proof_that_0.999..._equals_1#Proofs.2010.

[22] Knott, R. *Using Powers of Phi to Represent Integers (Base Phi)*. Available from: http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/phigits.html.2007.

[23] Stakhov, A., "Brousentsov's Ternary Principle, Bergman's Number System and Ternary Mirror-symmetrical Arithmetic," *The Computer Journal,* 45(2), 221-236(2002).

[24] Stakhov, A. *Ternary mirror-symmetrical representation*. Available from: http://www.goldenmuseum.com/1406Mirror_engl.html.2001.

[25] Ciet, M., et al., "Trading Inversions for Multiplications in Elliptic Curve Cryptography," *Designs, Codes and Cryptography,* 39(2), 189-206(2006).

[26] Hayes, B., "Third Base," *American Scientist,* 3(24(1908).

[27] Wikipedia. *Ternary numeral system*. Available from: http://en.wikipedia.org/wiki/Ternary_numeral_system.2010.

[28] Wikipedia. *Balanced ternary*. Available from: http://en.wikipedia.org/wiki/Balanced_ternary.2010.

[29] Museum, T. M. o. *Nikolay Brusentsov - the Creator of the Trinary Computer*. Available from: http://www.icfcst.kiev.ua/museum/Brusentsov.html.

[30] Gonzalez, R. C. and Woods, R. E., *Digital Image Processing*, 3 ed. Pearson Prentice Hall (2007).

[31] Govindarajani, B., et al., "Progressive Edge Detection on multi-bit images using polynomial-based binarization," *Machine Learning and Cybernetics, 2008 International Conference on*, 3714-3719(2008).
[32] Kamata, S., et al., "Depth-first coding for multivalued pictures using bit-plane decomposition," *IEEE Trans. Commun.,* 43(5), 1961-1969(1995).
[33] Loncar-Turukalo, T., et al., "Image compression by decomposition into bit planes," *Telecommunications in Modern Satellite, Cable and Broadcasting Service, 2001. TELSIKS 2001. 5th International Conference on*, 419-422 vol.2(2001).
[34] Yu, S. S. and Galatsanos, N. P., "Binary decompositions for high-order entropy coding of grayscale images," *IEEE Trans. Circuits Syst. Video Technol.,* 6(1), 21-31(1996).
[35] Han, J.-W., et al., "Optical image encryption based on XOR operations," *Opt. Eng.,* 38(1), 47-54(1999).
[36] Moon, D., et al., "An Efficient Selective Encryption of Fingerprint Images for Embedded Processors," *ETRI J.,* 28(4), 444-452(2006).
[37] Podesser, M., et al., "Selective bitplane encryption for secure transmission of image data in mobile environments," *The 5th Nordic Signal Processing Symposium -NORSIG-2002*, 1037(2002).
[38] Zheng, W., et al., "Image Data Encryption and Hiding Based on Wavelet Packet Transform and Bit Planes Decomposition," *Wireless Communications, Networking and Mobile Computing, 2008. WiCOM '08. 4th International Conference on*, 1-4(2008).
[39] Dey, S., et al., "An LSB Data Hiding Technique Using Prime Numbers," *2007 3rd Int. Sym. Information Assurance and Security*, 101-108(2007).
[40] Sun, Z. and Jiang, H., "An Adaptive Video Watermarking Based on Decomposing of Bit Planes," *Genetic and Evolutionary Computing, 2009. WGEC '09. 3rd International Conference on*, 324-326(2009).
[41] Noda, H., et al., "Application of bit-plane decomposition steganography to JPEG2000 encoded images," *IEEE Signal Process Lett.,* 9(12), 410-413(2002).
[42] Noda, H., et al., "Application of bit-plane decomposition steganography to wavelet encoded images," *2002 Int. Conf. Image Processing*, II-909-II-912 vol.2(2002).
[43] Panetta, K., et al., "Parameterized Logarithmic Framework for Image Enhancement," *IEEE Trans. Syst. Man Cybern. Part B Cybern.,* 41(2), 460-473(2011).